Turing Machines

Part Two

Outline for Today

- ***The Church-Turing Thesis***
  - Just how powerful are TMs?
- ***What Does it Mean to Solve a Problem?***
  - Rethinking what “solving” a problem means, and two possible answers to that question.

Recap from Last Time

Turing Machines

- A ***Turing machine*** is a program that controls a tape head as it moves around an infinite tape.
- There are six commands:
  - **`Move`** *direction*
  - **`Write`** *symbol*
  - **`Goto`** *label*
  - **`Return`** *boolean*
  - **`If`** *symbol command*
  - **`If Not`** *symbol command*
- Despite their limited vocabulary, TMs are surprisingly powerful.

A Sample Turing Machine

- Here's a sample TM.
- It receives inputs over the alphabet Σ = {**a**, **b**}.
- What strings does this TM accept?
- Can you write a regex that matches precisely the strings this TM accepts?

```
Start:
    If Not 'a' Return False

Loop:
    Move Right
    If Not Blank Goto Loop
    Move Left
    Move Left
    If Not 'b' Return False
    Return True
```

Answer at
https://pollev.com/cs103

What Can We Do With a TM?

- Last time, we saw TMs that
  - check if a string has the form $\mathtt{a}^n\mathtt{b}^n$,
  - check if a string has the same number of **a**'s and **b**'s and
  - sort a string of **a**'s and **b**'s.
- Here's a list of some other things TMs can do; check the starter files for PS8 to see them in action!
  - Check if a number is a Fibonacci number.
  - Convert the number *n* into a string of *n* **a**'s.
  - Check if a string is a *tautonym* (the same string repeated twice).
  - So much more!
- This hints at the idea that TMs might be more powerful than they look.

New Stuff!

Main Questions for Today:

Just how powerful are Turing machines?

What problems can you solve with a computer?

Real and "Ideal" Computers

- A real computer has memory limitations: you have a finite amount of RAM, a finite amount of disk space, etc.
- However, as computers get more and more powerful, the amount of memory available keeps increasing.
- An ***idealized computer*** is like a regular computer, but with unlimited RAM and disk space. It functions just like a regular computer, but never runs out of memory.

Theorem: Turing machines are equal in power to idealized computers. That is, any computation that can be done on a TM can be done on an idealized computer and vice-versa.

Key Idea: Two models of computation are equally powerful if they can simulate each other.

Simulating a TM

- The individual commands in a TM are simple and perform only basic operations:

```
Move   Write   Goto   Return   If
```

- The memory for a TM can be thought of as a string with some number keeping track of the current index.
- To simulate a TM, we need to
  - see which line of the program we're on,
  - determine what command it is, and
  - simulate that single command.
- ***Claim:*** This is reasonably straightforward to do on an idealized computer.
  - My "core" logic for the TM simulator is under fifty lines of code, including comments.

Simulating a TM

- Because a computer can simulate each individual TM instruction, a computer can do anything a TM can do.
- ***Key Idea:*** Even the most complicated TM is made out of individual instructions, and if we can simulate those instructions, we can simulate an arbitrarily complicated TM.

Simulating a Computer

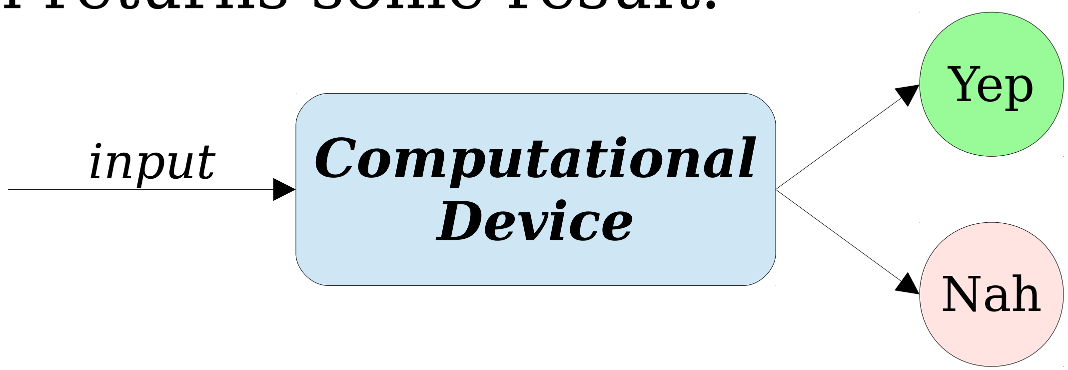
- Programming languages provide a set of simple constructs.
  - Think things like variables, arrays, loops, functions, classes, etc.
- You, the programmer, then combine these basic constructs together to assemble larger programs.
- ***Key Idea:*** A TM is powerful enough to simulate each of these individual pieces. It's therefore powerful enough to simulate anything a real computer can do.

A CS107 Perspective

- Internally, computers execute by using basic operations like
  - simple arithmetic,
  - memory reads and writes,
  - branches and jumps,
  - register operations,
  - etc.
- Each of these are simple enough that they could be simulated by a Turing machine.

A Leap of Faith

- ***Claim:*** A TM is powerful enough to simulate any computer program that gets an input, processes that input, then returns some result.



- The resulting TM might be colossal, or really slow, or both, but it would still faithfully simulate the computer.

This is quite a bold claim to make. What's an example of something your computer can do that you're convinced a Turing machine couldn't do?

Respond at https://pollev.com/cs103

Can a TM Work With…

"cat pictures?"

Sure! A picture is just a 2D array of colors, and a color can be represented as a series of numbers.

Can a TM Work With…

~~"cat pictures?"~~

"cat videos?"

If you think about it, a video is just a series of pictures!

Can a TM Work With…

"music?"

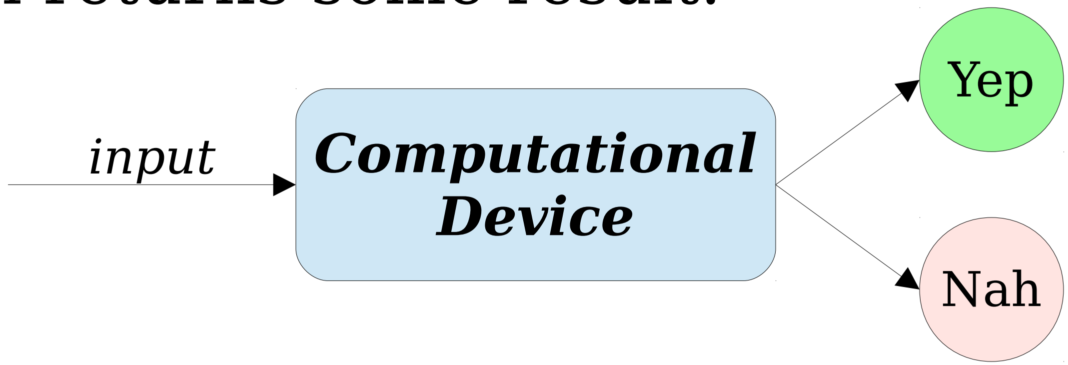
Sure! Music is encoded as a compressed waveform. That's just a list of numbers.

"deep learning?"

Sure! That's just applying a bunch of matrices and nonlinear functions to some input.

A Leap of Faith

- ***Claim:*** A TM is powerful enough to simulate any computer program that gets an input, processes that input, then returns some result.



- The resulting TM might be colossal, or really slow, or both, but it would still faithfully simulate the computer.
- We're going to take this as an article of faith in CS103. If you curious for more details, come talk to me after class.

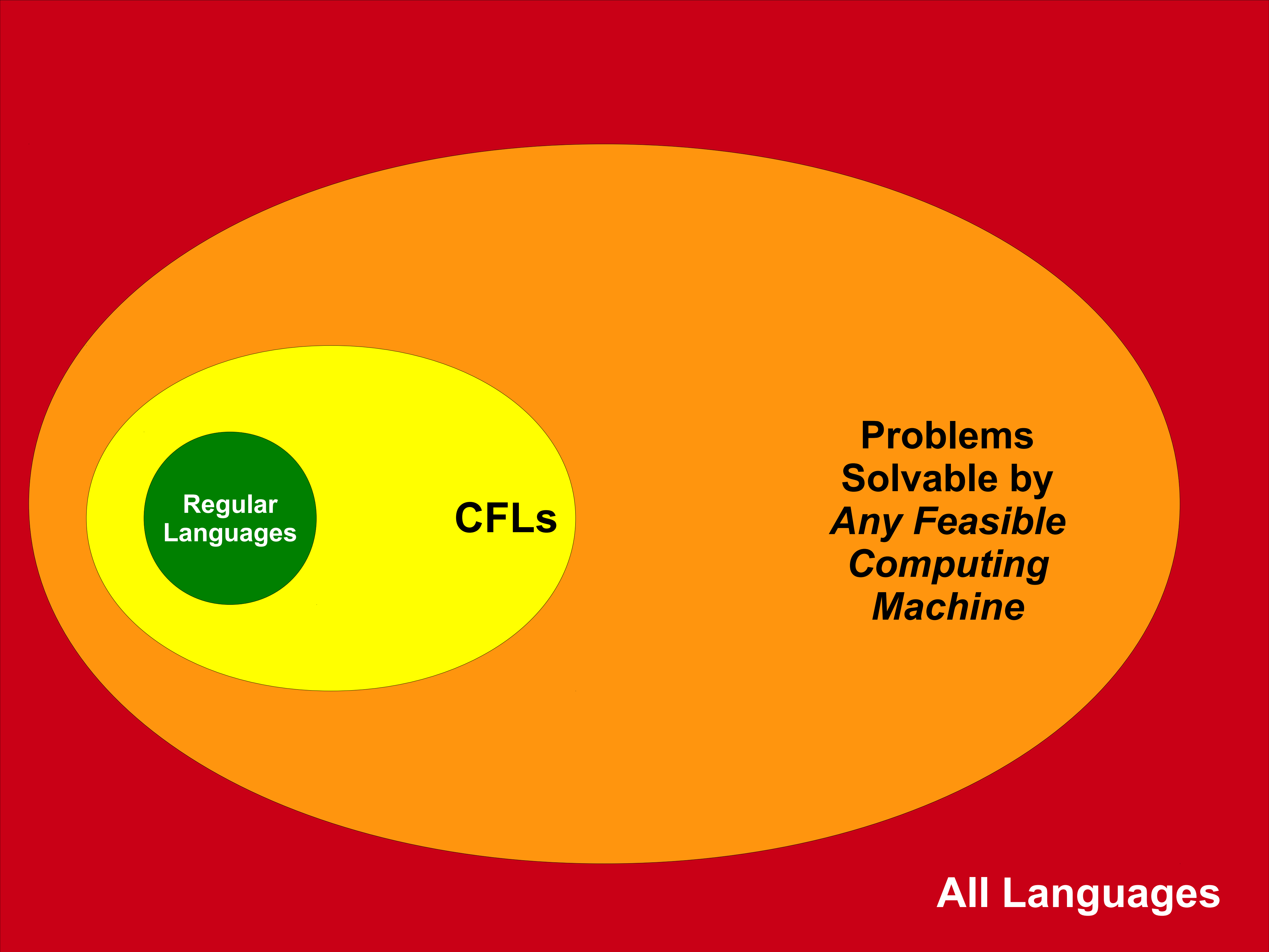
Just how powerful ***are*** Turing machines?

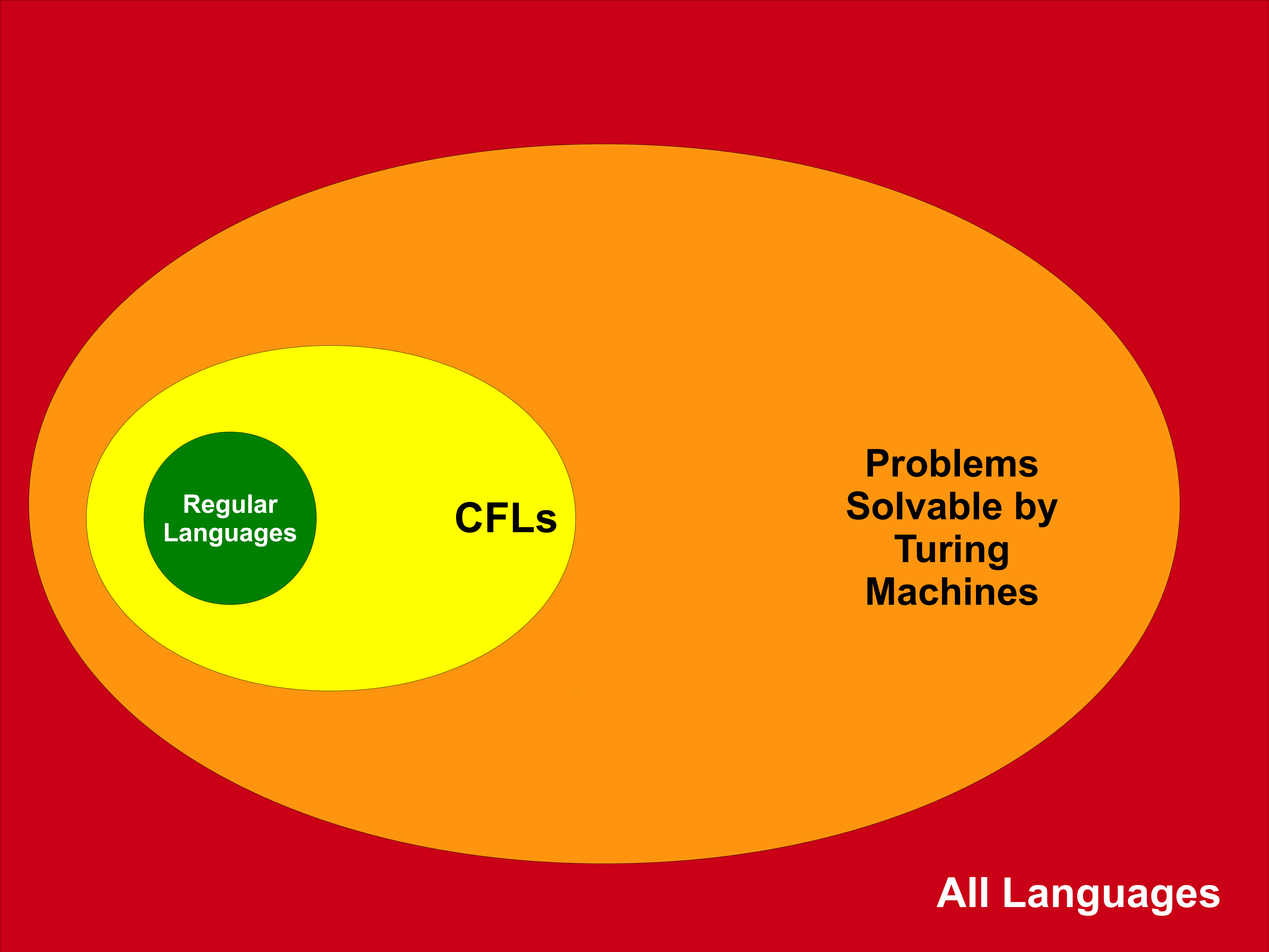
The ***Church-Turing Thesis*** claims that

every feasible method of computation is either equivalent to or weaker than a Turing machine.

“This is not a theorem – it is a falsifiable scientific hypothesis. And it has been thoroughly tested!”

- Ryan Williams


Regular Languages
CFLs
Problems Solvable by *Any Feasible Computing Machine*
All Languages


Regular Languages
CFLs
Problems Solvable by Turing Machines
All Languages

Time-Out for Announcements!

Problem Set 8

- If you turned in Problem Set Seven by 4:00PM today, we’ll apply a small on-time bonus to your score.
  - There is a 48 hour grace period which extends that deadline to 4:00PM on Sunday without penalty.
- Problem Set Eight goes out today. It’s due next Friday at 4:00PM.
  - Construct context-free grammars and explore their expressive power.
  - Dive deeper into the structure of languages and functions between languages.
  - Tinker with TMs and what it’s like to build all computation from smaller pieces.
- You know the drill: come talk to us if you have any questions, and let us know what we can do to help out.

Back to CS103!

Decidability and Recognizability

What problems can we solve with a computer?

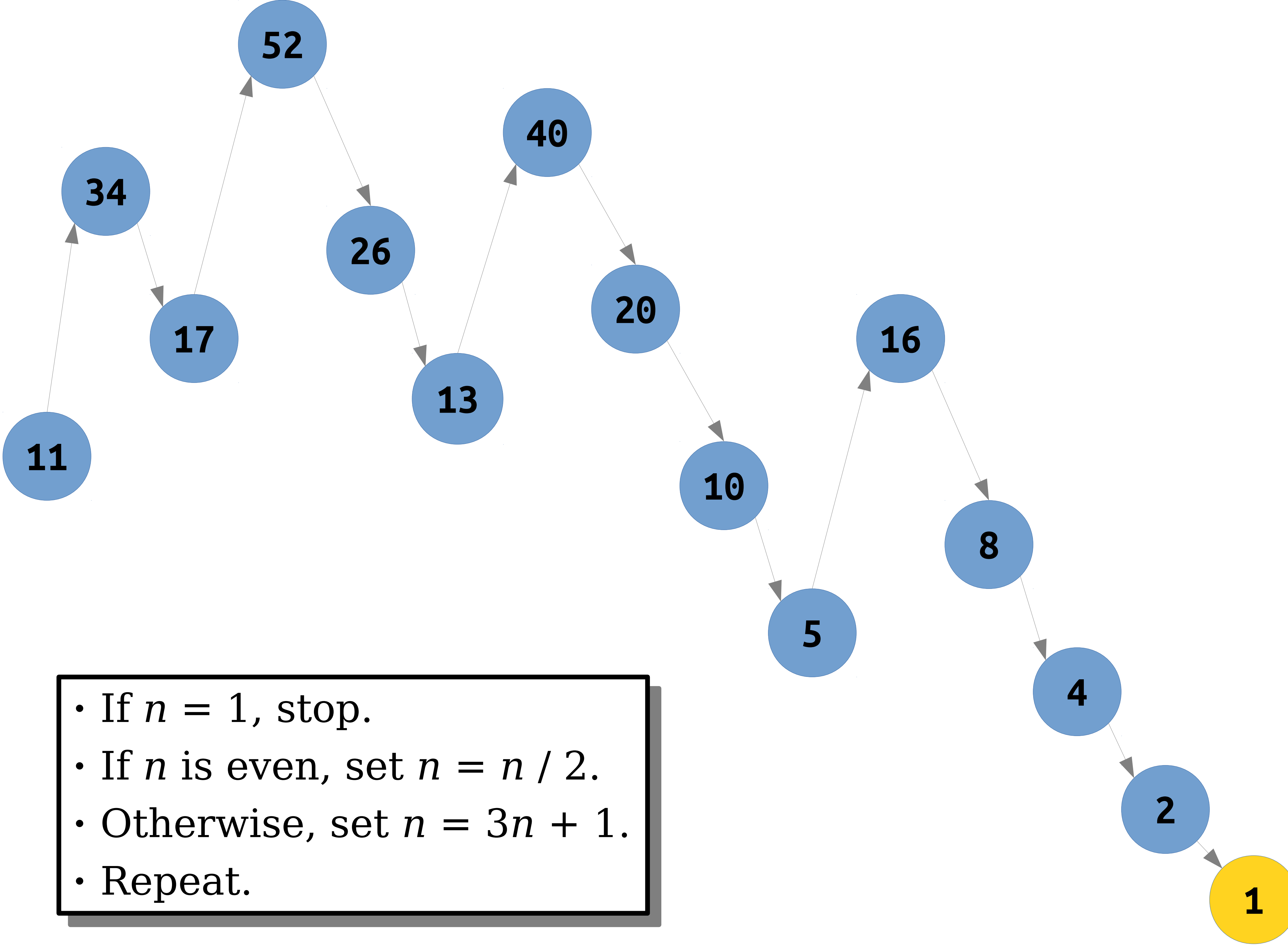
What kind of computer?

What problems can we solve with a computer?

What does it mean to "solve" a problem?

The Hailstone Sequence

- Consider the following procedure, starting with some $n \in \mathbb{N}$, where $n > 0$:
  - If $n = 1$, you are done.
  - If n is even, set $n = n / 2$.
  - Otherwise, set $n = 3n + 1$.
  - Repeat.
- ***Question:*** Given a natural number $n > 0$, does this process terminate?



- If $n = 1$, stop.
- If n is even, set $n = n / 2$.
- Otherwise, set $n = 3n + 1$.
- Repeat.

The Hailstone Sequence

- Consider the following procedure, starting with some $n \in \mathbb{N}$, where $n > 0$:
  - If $n = 1$, you are done.
  - If n is even, set $n = n / 2$.
  - Otherwise, set $n = 3n + 1$.
  - Repeat.
- Does the Hailstone Sequence terminate for…
  - $n = 5$?
  - $n = 20$?
  - $n = 7$?
  - $n = 27$?

Answer at
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The Hailstone Sequence

- Let $\Sigma = \{\mathtt{a}\}$ and consider the language

  $$L = \{ \mathtt{a}^n \mid n > 0 \text{ and the hailstone sequence terminates for } n \}.$$

- Could we build a TM for L?

The Hailstone Turing Machine

- We can build a TM that works as follows:
  - If the input is ε, reject.
  - While the string is not `a`:
    - If the input has even length, halve the length of the string.
    - If the input has odd length, triple the length of the string and append a `a`.
  - Accept.

Does this Turing machine accept all nonempty strings?

The Collatz Conjecture

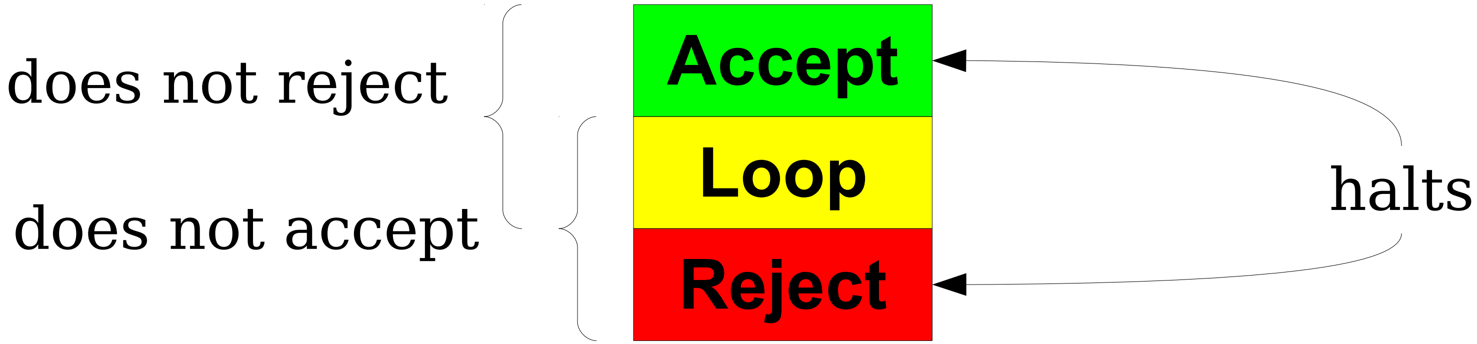
- It is *unknown* whether this process will terminate for all natural numbers.
- In other words, no one knows whether the TM described you just saw will always terminate!
- The conjecture (unproven claim) that the hailstone sequence always terminates is called the ***Collatz Conjecture***.
- This problem has eluded a solution for a long time. The influential mathematician Paul Erdős is reported to have said “Mathematics may not be ready for such problems.”

An Important Observation

- Unlike finite automata, which automatically halt after all the input is read, TMs keep running until they explicitly return true or return false.
- As a result, it's possible for a TM to run forever without accepting or rejecting.
- This leads to several important questions:
  - How do we formally define what it means to build a TM for a language?
  - What implications does this have about problem-solving?

Very Important Terminology

- Let M be a Turing machine.
- M ***accepts*** a string w if it returns true on w.
- M ***rejects*** a string w if it returns false on w.
- M ***loops infinitely*** (or just ***loops***) on a string w if when run on w it neither returns true nor returns false.
- M ***does not accept w*** if it either rejects w or loops on w.
- M ***does not reject w*** w if it either accepts w or loops on w.
- M ***halts on w*** if it accepts w or rejects w.



Recognizers and Recognizability

- A TM M is called a ***recognizer*** for a language L over Σ if the following statement is true:

$$\forall w \in \Sigma^*.\ (w \in L \leftrightarrow M \text{ accepts } w)$$

- A language L is called ***recognizable*** if there is a recognizer for it.
- If you are absolutely certain that $w \in L$, then running a recognizer for L on w will (eventually) confirm this.
  - Eventually, M will accept w.
- If you don't know whether $w \in L$, running M on w may never tell you anything.
  - M might loop on w – but you can't differentiate between "it'll accept if you wait longer" and "it will never come back with an answer."
- Does this feel like "solving a problem" to you?

Recognizers and Recognizability

- The hailstone TM M we saw earlier is a recognizer for the language

$$L = \{ \mathtt{a}^n \mid n > 0 \text{ and the hailstone sequence terminates for } n \}.$$

- If the sequence does terminate starting at n, then M accepts $\mathtt{a}^n$.
- If the sequence doesn't terminate, then M loops forever on $\mathtt{a}^n$ and never gives an answer.
- If you somehow knew the hailstone sequence terminated for n, this machine would (eventually) confirm this. If you didn't know, this machine might not tell you anything.

Recognizers and Recognizability

- Earlier this quarter you explored sums of four squares. Now, let's talk about sums of three cubes.
- Are there integers x, y, and z where…
  - $x^3 + y^3 + z^3 = 10$?
  - $x^3 + y^3 + z^3 = 11$?
  - $x^3 + y^3 + z^3 = 12$?
  - $x^3 + y^3 + z^3 = 13$?

Recognizers and Recognizability

- Surprising fact: until 2019, no one knew whether there were integers *x*, *y*, and *z* where

$$x^3 + y^3 + z^3 = 33.$$

- A heavily optimized computer search found this answer:

$$x = 8{,}866{,}128{,}975{,}287{,}528$$
$$y = -8{,}778{,}405{,}442{,}862{,}239$$
$$z = -2{,}736{,}111{,}468{,}807{,}040$$

- As of February 2024, no one knows whether there are integers *x*, *y*, and *z* where

$$x^3 + y^3 + z^3 = 114.$$

Recognizers and Recognizability

- Consider the language

$$L = \{\ \mathbf{a}^n \mid \exists x \in \mathbb{Z}.\ \exists y \in \mathbb{Z}.\ \exists z \in \mathbb{Z}.\ x^3 + y^3 + z^3 = n\ \}$$

- Here's pseudocode for a recognizer to see whether such a triple exists:

```
for max = 0, 1, 2, …
  for x from -max to +max:
    for y from -max to +max:
      for z from -max to +max:
        if x³ + y³ + z³ = n: return true
```

- If you somehow knew there was a triple x, y, and z where $x^3 + y^3 + z^3 = n$, running this program will (eventually) convince you of this.
- If you weren't sure whether a triple exists, this recognizer might not be useful to you.

Recognizers and Recognizability

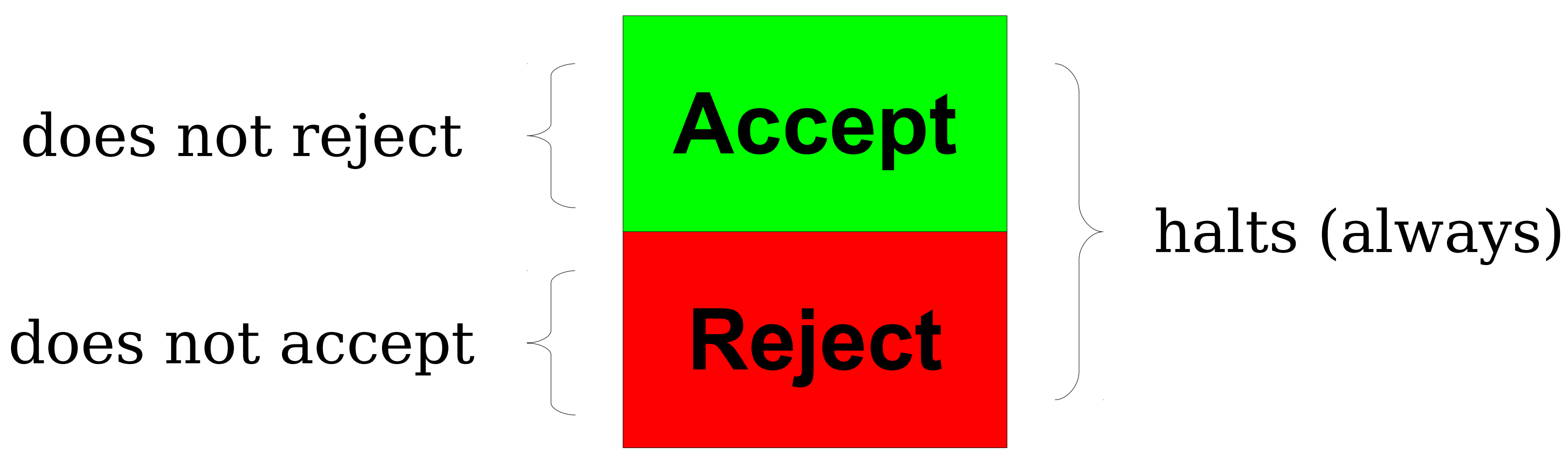
- The class **RE** consists of all recognizable languages.
- Formally speaking:

  $$\mathbf{RE} = \{\, L \mid L \text{ is a language and there's a recognizer for } L \,\}$$

- You can think of **RE** as "all problems with yes/no answers where "yes" answers can be confirmed by a computer."
  - Given a recognizable language L and a string $w \in L$, running a recognizer for L on w will eventually confirm $w \in L$.
  - The recognizer will never have a "false positive" of saying that a string is in L when it isn't.
- This is a "weak" notion of solving a problem.
- Is there a "stronger" one?

Deciders and Decidability

- Some, but not all, TMs have the following property: the TM halts on all inputs.
- If you are given a TM M that always halts, then for the TM M, the statement "M does not accept w" means "M rejects w."



Deciders and Decidability

- A TM M is called a ***decider*** for a language L over Σ if the following statements are true:

$$\forall w \in \Sigma^*.\ M \text{ halts on } w.$$

$$\forall w \in \Sigma^*.\ (w \in L \quad \leftrightarrow \quad M \text{ accepts } w)$$

- A language L is called ***decidable*** if there is a decider for it.
- A decider M for a language L accepts all strings in L and rejects all strings not in L.
- A decider M for a language L is a recognizer for L that halts on all inputs.
- Intuitively, if you don't know whether $w \in L$, running M on w will "create new knowledge" by telling you the answer.
- This is a "strong" notion of "solving a problem."

Deciders and Decidability

- While no one knows whether there are integers x, y, and z where

  $$x^3 + y^3 + z^3 = 114,$$

  it is very easy to figure out whether there are integers x, y, and z where

  $$x^2 + y^2 + z^2 = 114.$$

- Why is this?

Deciders and Decidability

- Consider the language

  $L = \{ \mathtt{a}^n \mid \exists x \in \mathbb{Z}.\ \exists y \in \mathbb{Z}.\ \exists z \in \mathbb{Z}.\ x^2 + y^2 + z^2 = n \}.$

- Here's pseudocode for a decider to see whether such a triple exists:

```
for x from 0 to n:
  for y from 0 to n:
    for z from 0 to n:
      if x² + y² + z² = n: return true
return false
```

- After trying all possible options, this program will either find a triple that works or report that none exists.

Deciders and Decidability

- The class **R** consists of all decidable languages.
- Formally speaking:

  R = { L | L is a language and there's a decider for L }

- You can think of **R** as "all problems with yes/no answers that can be fully solved by computers."
  - Given a decidable language, run a decider for L and see what happens.
  - Think of this as "knowledge creation" – if you don't know whether a string is in L, running the decider will, given enough time, tell you.
- The class **R** contains all the regular languages, all the context-free languages, most of CS161, etc.
- This is a "strong" notion of solving a problem.

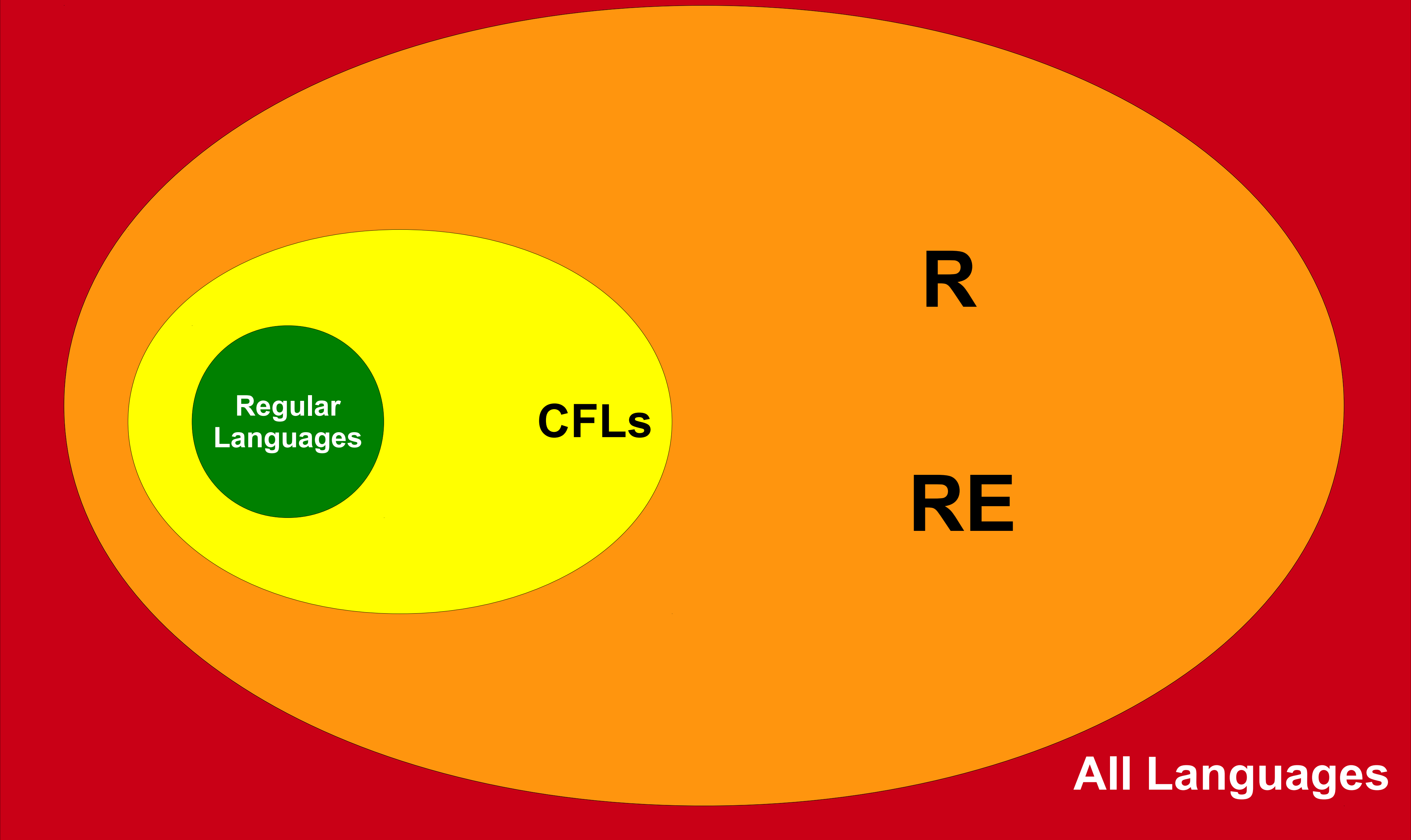
R and **RE** Languages

- Every decider for L is also a recognizer for L.
- This means that $\mathbf{R} \subseteq \mathbf{RE}$.
- Hugely important theoretical question:

$$\mathbf{R} \stackrel{?}{=} \mathbf{RE}$$

- That is, if you can just confirm "yes" answers to a problem, can you necessarily *solve* that problem?

Which Picture is Correct?



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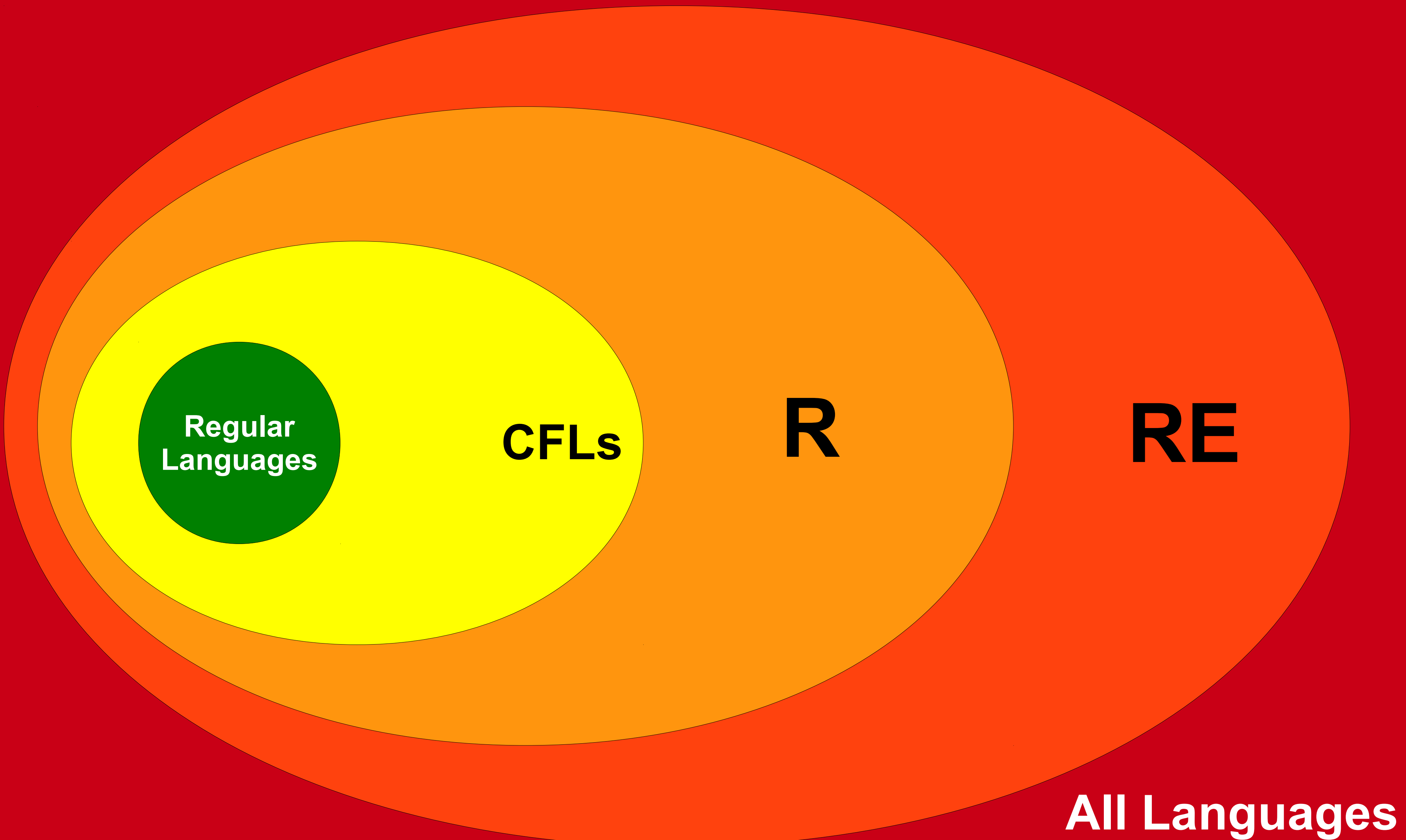

Unanswered Questions

- Why exactly is **RE** an interesting class of problems?
- What does the $\mathbf{R} \stackrel{?}{=} \mathbf{RE}$ question mean?
- Is $\mathbf{R} = \mathbf{RE}$?
- What lies beyond **R** and **RE**?
- We'll see the answers to each of these in due time.

Next Time

- ***Emergent Properties***
  - Larger phenomena made of smaller parts.
- ***Universal Machines***
  - A single, “most powerful” computer.
- ***Self-Reference***
  - Programs that ask questions about themselves.